WEEK 3: COMPLEX FOURIER AND LAPLACE INTEGRALS
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1. COMPLEX INTEGRALS

Previously, we saw that for real amplitude $A(x)$ and phase $\phi(x)$, we have the expansion

$$\int_0^{\epsilon} A(x) \exp(-\lambda\phi(x)dx \sim \sum_{j=0}^{\infty} a_j \lambda^{-(1+j)/k},$$

where

$$A(x) = \sum_{j=l}^{\infty} b_j x^j$$
$$\phi(x) = \sum_{j=k}^{\infty} c_j x^j,$$

and a_j is a polynomial in $b_l, \dots, b_j, c_k^{-1}, c_k, \dots, c_j$. For complex $A(z)$, the extension of the result is trivial as we can write

$$A(z) = \Re\{A\}(z) + i\Im\{A\}(z),$$

and apply the above result to each summand.

The real work lies in extending ϕ to the complex plane. Suppose that A and ϕ are analytic in a neighbourhood of the origin (so that we may deform our contour of integration without worry).

1.1. **Step 1 – Evaluation of the One-Sided Integral.** Recall that last class we had defined

$$I_+(\lambda) = \int_{\gamma_+} A(z) \exp(-\lambda\phi(z))dz,$$

where γ_+ is the restriction of our curve γ to the domain $[0, \epsilon]$. We will evaluate I_+ by making the change of variables $y = \phi(z)^{1/k}$. Care must be taken in choosing the k^{th} root, as the expression

$$y = c_k^{1/k} x \left(1 + \frac{c_{k+1}}{c_k}x + \cdots + \frac{c_M}{c_k}x^{M-k} + O(x^{M-k+1})\right)^{1/k},$$

which we used previously, now defines k different functions, each analytic in a neighbourhood of the origin. Because of this, we define the *primitive k^{th} root* by the analytic function

$$\mathfrak{p} : \mathbb{C} \setminus \mathbb{R}_{<0} \to K = \{z : -\pi/k < \arg(z) < \pi/k\},$$

where $\mathfrak{p}(u^{1/k})$ is the unique $z \in K$ such that $z^k = u$.

Now, let $\Upsilon = C\gamma'(0)$, for some positive real number C. From our previous results we know $\phi(x) \sim c_k z^{1/k}$, and by assumption $\Re\{\phi(\gamma(t))\} \geq 0$, which forces Υ to be in the set of pre-images of the positive real half-plane under the mapping $c_k z^k$.

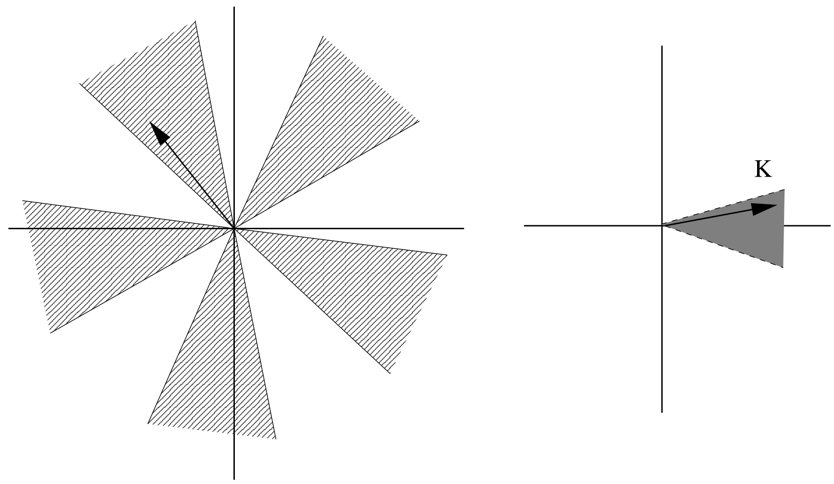

FIGURE 1. The quantity Υ and its image under $c_k z^k$ in K.

Define $f(x) := \mathfrak{p}(\phi(x)^{1/k})$. Since $\phi(\gamma(t))$ remains in the positive real half-plane, it must always remain in the slit plane, and thus $f(\gamma_+(t)) \subset K$. The change of variables described above becomes

$$y = f(x) = \eta x \left(1 + \frac{c_{k+1}}{c_k} x + \cdots + \frac{c_M}{c_k} x^{M-k} + O(x^{M-k+1})\right)^{1/k},$$

where $\eta = \Upsilon^{-1}\mathfrak{p}((c_k \Upsilon^k)^{1/k})$ and y fixes 1 (since $\mathfrak{p}(1) = 1$). The Chain Rule implies

$$\eta = f'(0) \qquad\qquad \left.\frac{d}{dt} f(\gamma_+(t))\right|_{t=0} = \eta\Upsilon,$$

and the inverse $x = g(y)$ is given by taking $c_k^{1/k}\eta$ in our expansion from last week:

$$x = \sum_{j=1}^{M-k} e_j \left(\frac{y}{\eta}\right)^j + O(y^{M-k+1}),$$

where e_j is a polynomial in $c_{k+1}, \ldots, c_j$ which can be made explicit. As in the real case, we now have the expression

$$I_+(\lambda) = \int_{\tilde{\gamma}} \tilde{A}(y) \exp(-\lambda y^k) dy,$$

where $\tilde{\gamma} = f \circ \gamma_+$.

Let p be the endpoint of $\tilde{\gamma}$ at $t = \epsilon$, $p' = \Re(p)$, α be the line segment from the origin to p', and β be the line segment from p' to p.

Then $\tilde{\gamma}$ is homotopic to $\alpha + \beta$, the curve obtained by joining the end of α to the beginning of β, and for a function $h(z)$ with analyticity conditions the same as our integrand above

$$\int_{\tilde{\gamma}} h(z)dz = \int_\alpha h(z)dz + \int_\beta h(z)dz.$$

On a compact subset of K, $\Re\{y^k\}$ is bounded below by a positive constant. Thus, on $\beta \subset K$ there exists constants C and θ such that

$$\left|\tilde{A}\exp(-\lambda y^k)\right| \leq C e^{-\theta\lambda},$$

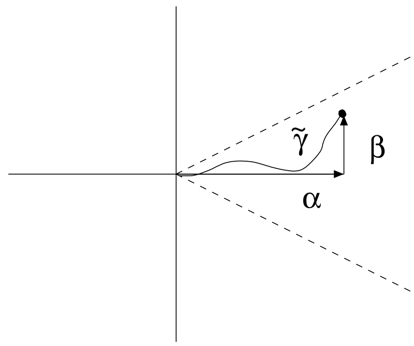

FIGURE 2. The line segments α and β.

and

$$I_+ = \int_\alpha \tilde{A}(y)\exp(-\lambda y^k)dy + R,$$

where $R \to 0$ exponentially. Applying our previous result to

$$\int_\alpha \tilde{A}(y)\exp(-\lambda y^k)dy$$

implies

$$I_+(\lambda) \sim \sum_{j=l}^{\infty} a_j C(k,j) c_k^{-(1+j)/k} \lambda^{-(1+j)/k}.$$

1.2. **Step 2 – Two sided integrals.** In this section, we consider the expansion of the integral

$$I(\lambda) = \int_\gamma A(z)\exp(-\lambda\phi(z))dz,$$

where $\gamma : [-\epsilon, \epsilon] \to \mathbb{C}$. To reduce this to the 1-sided integral, we define the contour $\gamma_- : [0, \epsilon] \to \mathbb{C}$ by $t \mapsto \gamma(-t)$. As we have changed direction, we have $I = I_+ - I_-$, where I_- is the integral along γ_-.

The only difference between I_+ and I_- is the difference between the curves γ_+ and γ_-, which affects our choice of η, however it must be the case that $\eta_- = \eta_+/\omega$ for some $\omega^k = 1$. Let the inverses for $y = \phi(x)^{1/k}$ be given by $g_+(y)$ and $g_-(y)$ on the two curves, so $g_-(y) = g_+(\omega y)$. We also produce amplitudes $\tilde{A}_+(y)$ and $\tilde{A}_-(y)$ satisfying

$$\begin{aligned}
\tilde{A}_+(y) &= A(g_+(y)) \cdot g_+'(y) \\
\tilde{A}_-(y) &= A(g_-(y)) \cdot g_-'(y) \\
&= A(g_-(\omega y)) \cdot g_-'(\omega y) \\
&= \omega \tilde{A}_+(\omega y).
\end{aligned}$$

Putting this together, we get $[y^j]\tilde{A}_- = \omega^{j-1}[y^j]\tilde{A}_+$, and integrating term-by-term followed by taking the asymptotic expansion of $\tilde{A}_+$ tells us $\alpha_j = (1 - \omega^{j+1})a_j$, where the a_j are the coefficients of $I_+(\lambda)$. There are two cases to consider:

(a) When k is even: Since $\phi(z) \sim c_k z^k$, $\phi(\gamma(t))$ does a U-turn at the origin, with the tangents to $\phi(\gamma_-(t))$ and $\phi(\gamma_+(t))$ coinciding there.

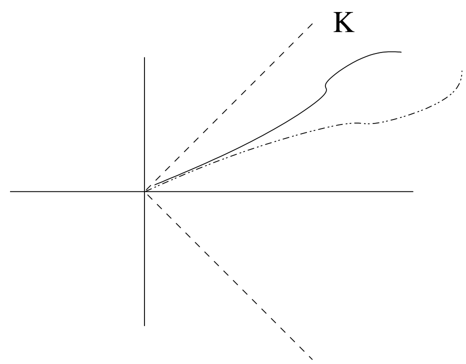

FIGURE 3. The case when k is even.

As γ_- reverses orientation, $\Upsilon_- = \gamma'(0) = -\Upsilon_+$ and

$$\eta_- = \Upsilon_-^{-1}\mathfrak{p}((c_k\Upsilon_-^k)^{1/k}) = -\,\Upsilon_+^{-1}\,\mathfrak{p}((c_k\Upsilon_+^k)^{1/k}) = -\eta_+,$$

since k is even. This implies $\omega = -1$ and

$$\alpha_j = \begin{cases} 2a_j & : j \text{ is even} \\ 0 & : o.w. \end{cases} .$$

(b) <u>When k is odd:</u> The images of $\Im\{\phi(\gamma_-(t))\}$ and $\Im\{\phi(\gamma_+(t))\}$ have opposite signs and point in opposite directions.

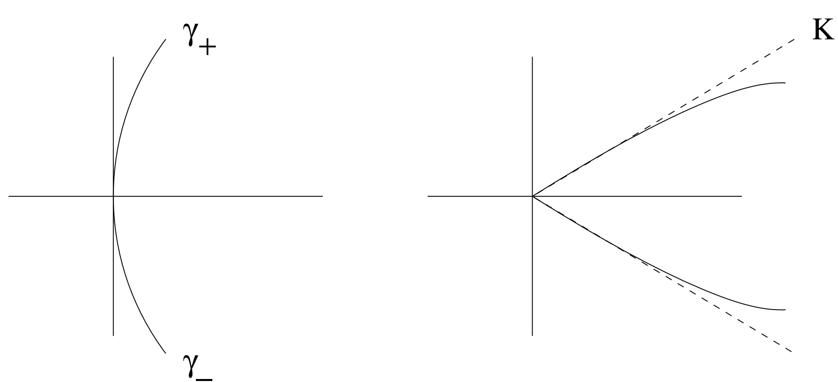

FIGURE 4. The case when k is odd.

Thus, the argument of the tangent to $\phi(\gamma_+(t))$ is $\sigma\pi/2$ while the argument of $\phi(\gamma_-(t))$ is $-\sigma\pi/2$, where $\phi = \operatorname{sgn}(\Im\{\phi(\gamma'(0))\})$. The arguments differ by $-\sigma\pi$ and shrink by a factor of k when we take the k^{th} root so again we find that $\Upsilon_- = -\Upsilon_+$, while now $\eta_- = \eta_+/\omega$ where $\omega = -e^{i\pi\sigma/k}$.

2. WATSON'S LEMMA

This machinery allows us to prove Watson's Lemma:

Proposition 1. *Let $A : \mathbb{R}^+ \to \mathbb{C}$ have asymptotic expansion*

$$A(t) = \sum_{m=0}^{\infty} b_m t^{\beta_m},$$

with $-1 < \Re\{\beta_0\} < \Re\{\beta_1\} < \cdots$ and $\Re\{\beta_m\} \to \infty$. Then

$$L(\lambda) := \int_0^\infty A(t) e^{-\lambda t} dt \sim \sum_{m=0}^{\infty} b_m \Gamma(\beta_m) \lambda^{-(1+j)/k}.$$

Proof. Truncating the path in the integral from the positive real line to $[0, \epsilon]$ introduces only an exponentially small error. Writing

$$A(t) = \sum_{m=0}^{N} b_m t^{\beta_m} + R_N(t),$$

with $R_N(t) = O\left(t^{\Re\{\beta_{N+1}\}}\right)$ at 0, we plug the above into $L(\lambda)$ and integrate term by term to get

$$L(\lambda) \sim \sum_{m=0}^{N} \int_0^\epsilon b_m t^{\beta_m} e^{-\lambda t} dt + \int_0^\epsilon R_N(t) e^{-\lambda t} dt.$$

Our previous 'Big-O Lemma' (Lemma 4.2.1 in the text) then implies

$$\left| \int_0^\epsilon R_N(t) e^{-\lambda t} dt \right| \to 0,$$

and the finite integrals can be evaluated as Laplace transforms. □

3. A Partial Example

We conclude by a (partial) example of how this method can be utilized. Let

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xt + t^3/3)} dt,$$

be the Airy function – we will outline how one can determine the asymptotics of $Ai(x)$ as $x \to \infty$.

First, we change variables by setting $t = i\sqrt{x}u$, which implies

$$Ai(x) = \frac{i\sqrt{x}}{2\pi} \int_{i\mathbb{R}} e^{-x^{3/2}(u - u^3/3)} du,$$

and hence we have $A(u) = 1$ and $\phi(u) = u - u^3/3$ here, with respect to the results above. This gives critical points $u = \pm 1$, where we have the Taylor expansions:

$$T^+ = \frac{2}{3} - (u-1)^2 - \frac{1}{3}(u-1)^3$$
$$T^- = \frac{-4}{3} + (u+1)^2 - \frac{1}{3}(u+1)^3.$$

As we care only about dominant term asymptotics, we truncate the expansions $T^+(u) - T^+(0)$ and $T^-(u) - T^-(0)$ to second order and parametrize their real positive solutions by

$$y^2 = T^+(u) - T^+(0)$$
$$y_-^2 = T^-(u) - T^-(0).$$

Now consider the section of the integral near $u = 1$ (one can deal with the integral near $u = -1$ analagously). Here we have $y^2 = -(u-1)^2$ which gives a parametrization $y = i(u-1)$. Thus, the contribution of the full integral near $u = 1$, $I(\lambda)$, is given by

$$
\begin{aligned}
I(\lambda) &\sim -i\int_{1-\epsilon}^{1+\epsilon} e^{-\lambda((1-iy)-(1-iy)^3/3)}dy \\
&= -ie^{-\lambda^{2/3}}\int_{1-\epsilon}^{1+\epsilon} e^{-\lambda(y^2-iy^3/3)}dy \\
&= -ie^{-\lambda^{2/3}}\int_{1-\epsilon}^{1+\epsilon} e^{-\lambda y^2}\sum_{n=0}^{\infty}\frac{(i\lambda y^3/3)^n}{n!}dy \\
&= -ie^{-\lambda^{2/3}}\sum_{n=0}^{\infty}\int_{1-\epsilon}^{1+\epsilon}\frac{e^{-\lambda y^2}(i\lambda y^3/3)^n}{n!}dy,
\end{aligned}
$$

and one may now calculate the above integrals. Applying the same approach to the contribution of the full integral near $u = -1$ allows one to recover the asymptotics of $Ai(x)$.